On the energy-momentum tensor in non-commutative gauge theories

Ashok Das[a] and J. Frenkel[b]

[a] *Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA and*
[b] *Instituto de Física, Universidade de São Paulo, São Paulo, SP 05315-970, BRAZIL*

We study the properties of the energy-momentum tensor in non-commutative gauge theories by coupling them to a weak external gravitational field. In particular, we show that the stress tensor of such a theory coincides exactly with that derived from a theory where a Seiberg-Witten map has been implemented (namely, the procedure is commutative). Various other interesting features are also discussed.

PACS numbers: 11.15.-q

arXiv:hep-th/0212122 v1 11 Dec 2002

Non-commutative field theories [1, 2] have been of much interest in recent years following developments in superstring theory. In such theories, the natural parameter of non-commutativity, $\theta^{\mu\nu}$, is a dimensionful quantity. It is assumed to be a constant anti-symmetric tensor and this leads to a violation of Lorentz invariance in such theories, which also puts a bound on the magnitude of the parameter of non-commutativity [3]. In view of these, it is expected that scale invariance is also violated and correspondingly, the energy-momentum tensor has been studied in non-commutative gauge theories from several points of view [4, 5, 6, 7].

In non-commutative gauge theories, there is a map known as the Seiberg-Witten map [1], which ensures the gauge equivalence between the non-commutative theory and that of an ordinary gauge theory defined on a commutative space. A surprising outcome of these studies [5], which use an improved Belinfante method [8], is that the energy momentum tensor of the non-commutative theory, upon using the Seiberg-Witten map, does not coincide with the energy-momentum tensor of the theory obtained from this map. Namely, the Seiberg-Witten map and the derivation of the energy-momentum tensor do not seem to commute. In this brief report, we study this question by coupling the non-commutative gauge theory (with and without the Seiberg-Witten map) to gravity [9, 10, 11]. We show that the energy-momentum tensors derived by this method do coincide, namely the map commutes with the derivation of the energy-momentum tensor. The energy-momentum tensors are shown to be manifestly symmetric and traceless, but conserved covariantly. It is shown that it is possible to obtain, from this, a conserved tensor which, however, is neither symmetric and traceless and coincides with the one obtained from the improved Belinfante method. We shall discuss here, for simplicity, only the non-commutative $U(1)$ gauge theory, although the above conclusions naturally extend to the $U(N)$ case.

The non-commutative $U(1)$ theory is described by the gauge invariant action

$$S = -\frac{1}{4}\int d^4x\, \hat{F}_{\mu\nu}\star\hat{F}^{\mu\nu} \tag{1}$$

where the star product of two functions is defined by

$$A(x)\star B(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^\eta\partial_\nu^\xi}\, A(x+\eta)B(x+\xi)|_{\eta=\xi=0} \tag{2}$$

and the field strength tensor has the form

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i(\hat{A}_\mu\star\hat{A}_\nu - \hat{A}_\nu\star\hat{A}_\mu) \tag{3}$$

where $\hat{A}_\mu$ denotes the $U(1)$ gauge field defined on a non-commutative space. The Euler-Lagrange equations following from (1) lead to

$$\hat{D}_\mu\star\hat{F}^{\mu\nu} = \partial_\mu\hat{F}^{\mu\nu} - i(\hat{A}_\mu\star\hat{F}^{\mu\nu} - \hat{F}^{\mu\nu}\star\hat{A}_\mu) = 0 \tag{4}$$

By coupling the theory in (1) to a weak external gravitational field, we obtain

$$S = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,\star g^{\mu\lambda}\star g^{\nu\rho}\star\hat{F}_{\mu\nu}\star\hat{F}_{\lambda\rho} \tag{5}$$

where g denotes the determinant of the metric with a signature $(+,-,-,-)$. The energy-momentum tensor of the theory can now be obtained as

$$\hat{T}_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}} \tag{6}$$

We note that the star product involving the metric is not of consequence since the metric is set to the Minkowski metric at the end. It follows now in a straight forward manner that, for the theory in (1),

$$\hat{T}_{\mu\nu} = -\frac{1}{2}\left(\hat{F}_{\mu\lambda} \star \hat{F}_{\nu}^{\ \lambda} + \hat{F}_{\nu\lambda} \star \hat{F}_{\mu}^{\ \lambda} - \frac{1}{2}\,\eta_{\mu\nu}\,\hat{F}_{\lambda\rho} \star \hat{F}^{\lambda\rho}\right) \tag{7}$$

This tensor is manifestly symmetric and traceless and coincides with the one obtained through the conventional derivation in [5]. Furthermore, using the equations of motion (4) as well as Bianchi identity, it can be checked that this tensor is covariantly conserved, namely,

$$\hat{D}_{\mu} \star \hat{T}^{\mu\nu} = 0 \tag{8}$$

In this theory, the Seiberg-Witten map, to leading order in $\theta^{\mu\nu}$, leads to the correspondence

$$\begin{aligned} \hat{A}_{\mu} &= A_{\mu} - \frac{1}{2}\,\theta^{\alpha\beta}A_{\alpha}(\partial_{\beta}A_{\mu} + F_{\beta\mu}) + O(\theta^2) \\ \hat{F}_{\mu\nu} &= F_{\mu\nu} - \theta^{\alpha\beta}(F_{\mu\alpha}F_{\beta\nu} + A_{\alpha}\partial_{\beta}F_{\mu\nu}) + O(\theta^2) = F_{\mu\nu} - ((F\theta F)_{\mu\nu} + (A\theta\partial)F_{\mu\nu}) + O(\theta^2) \end{aligned} \tag{9}$$

where we have used an obvious matrix notation for simplicity. Here, the variables without a "hat" correspond to quantities defined on a commutative space and correspondingly

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{10}$$

Substituting the map (9) into (1), we obtain, to order θ,

$$S = -\frac{1}{4}\int d^4x\,\left[(1 - \frac{1}{2}\,\theta^{\alpha\beta}F_{\alpha\beta})\,F_{\mu\nu}F^{\mu\nu} + 2\mathrm{Tr}\,(\theta F^3) - \partial_{\beta}\left(\theta^{\alpha\beta}A_{\alpha}F_{\mu\nu}F^{\mu\nu}\right)\right] \tag{11}$$

Although a total divergence does not contribute to the equations of motion and is normally neglected in flat space-time, it does contribute to the energy-momentum tensor [9] and that is the reason why we have manifestly kept such a term in (11). The Euler-Lagrange equations following from this action lead to

$$\partial_{\mu}\left[(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})\,F^{\mu\nu} - ((F^2\theta)^{\mu\nu} + (F\theta F)^{\mu\nu} + (\theta F^2)^{\mu\nu}) - \frac{1}{4}\,\theta^{\mu\nu}\,F_{\lambda\rho}F^{\lambda\rho}\right] = 0 \tag{12}$$

Furthermore, by coupling (11) to a weak external gravitational field, we can derive the energy-momentum tensor, as in (6), to obtain

$$\begin{aligned} T_{\mu\nu} &= -(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})\left(F_{\mu\lambda}F_{\nu}^{\ \lambda} - \frac{1}{4}\,\eta_{\mu\nu}\,F_{\lambda\rho}F^{\lambda\rho}\right) - \left((F\theta F^2)_{\mu\nu} + (F\theta F^2)_{\nu\mu} - \frac{1}{2}\,\eta_{\mu\nu}\,\mathrm{Tr}(F\theta F^2)\right) \\ &\quad + \partial_{\beta}\left(\theta^{\alpha\beta}A_{\alpha}(F_{\mu\lambda}F_{\nu}^{\ \lambda} - \frac{1}{4}\,\eta_{\mu\nu}\,F_{\lambda\rho}F^{\lambda\rho})\right) \end{aligned} \tag{13}$$

We note that the energy-momentum tensor is manifestly symmetric and traceless as is $\hat{T}_{\mu\nu}$ in (7). Furthermore, it can be easily checked that, under the Seiberg-Witten map in (9) to leading order in θ,

$$\hat{T}_{\mu\nu} = T_{\mu\nu} \tag{14}$$

Namely, the energy-momentum tensor of the non-commutative theory, under a Seiberg-Witten map, goes over to that obtained from a theory with the map so that the process is commutative. Furthermore, under a Seiberg-Witten map to the leading order in θ, eq. (4) gives

$$\partial_{\mu}\left[(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})F^{\mu\nu} - (F\theta F)^{\mu\nu} - (\theta F^2)^{\mu\nu}\right] = 0 \tag{15}$$

The dynamical equations (12) and (15) do not seem to coincide at first sight. However, it is easy to check, with the help of the relation (which holds to linear order in θ),

$$\partial_{\mu}\left[(F^2\theta)^{\mu\nu} + \frac{1}{4}\,\theta^{\mu\nu}\,F_{\lambda\rho}F^{\lambda\rho}\right] = 0 \tag{16}$$

that the dynamical equations following from the non-commutative theory go over, under a Seiberg-Witten map, to the ones following from the deformed theory under the same map.

Since the dynamical equations as well as the energy-momentum tensors of the two theories are the same (to linear order in θ), it follows from (8) that

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left[(\theta F T^{(0)})^{\mu\nu} - \partial_\beta(\theta^{\alpha\beta} A_\alpha T^{(0)\,\mu\nu})\right] \tag{17}$$

where $T^{(0)\,\mu\nu}$ corresponds to the energy-momentum tensor with $\theta = 0$. Thus, we see that the energy-momentum tensor of the Seiberg-Witten deformed theory is not conserved in the ordinary sense, which is a reflection of the covariant conservation of the stress tensor in the non-commutative theory. However, it is clear that we can define a modified energy-momentum tensor

$$\begin{aligned}\overline{T}^{\mu\nu} &= T^{\mu\nu} - (\theta F T^{(0)})^{\mu\nu} + \partial_\beta(\theta^{\alpha\beta} A_\alpha T^{(0)\,\mu\nu}) \\ &= -(1 - \frac{1}{2}\theta^{\alpha\beta} F_{\alpha\beta})(F^{\mu\lambda} F^\nu_\lambda - \frac{1}{4}\,\eta^{\mu\nu}\, F_{\lambda\rho} F^{\lambda\rho}) - \Big((F\theta F^2)^{\mu\nu} + (F\theta F^2)^{\nu\mu} - \frac{1}{2}\eta^{\mu\nu}\mathrm{Tr}\,(F\theta F^2)\Big) \\ &\quad -(\theta F T^{(0)})^{\mu\nu}\end{aligned} \tag{18}$$

which will be conserved, namely,

$$\partial_\mu \overline{T}^{\mu\nu} = 0 \tag{19}$$

However, this tensor is no longer symmetric or traceless because of the last term in (18). Furthermore, it coincides with the energy-momentum tensor derived from the improved Belinfante method.

Acknowledgment:

This work was supported in part by US DOE Grant number DE-FG 02-91ER40685, by CNPq and FAPESP, Brasil.

[1] N. Seiberg and E. Witten, "String theory and non-commutative geometry", JHEP **09**, 032 (1999).

[2] M. R. Douglas and N. A. Nekrasov, "Non-commutative field theory", Rev. Mod. Phys. **73**, 977 (2002); R. J. Szabo, "Quantum field theory on non-commutative spaces", hep-th/0109162.

[3] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, "Non-commutative field theory and Lorentz violation", Phys. Rev. Lett. **87**, 141601 (2001).

[4] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Non-commutative Lorentz symmetry and the origin of the Seiberg-Witten map", Eur. Phys. Jour. **C24**, 165 (2002).

[5] J. M. Grimstrup, B. Kloiböck, L. Popp, V. Putz, M. Schewda and M. Wickenhauser, "The energy-momentum tensor in non-commutative gauge field models", hep-th/0210288.

[6] S. I. Kruglov, "Maxwell's theory on non-commutative spaces and quaternions", Annal. Fond. Broglie **27**, 343 (2002).

[7] M. Abou-Zeid and H. Dorn, "Comments on the energy-momentum tensor in non-commutative field theories", Phys. Lett. **B514**, 183 (2001).

[8] R. Jackiw, "Gauge covariant conformal transformations", Phys. Rev. Lett. **41**, 1635 (1978).

[9] A. Das, "Stress tensor in a class of gauge theories", Phys. Rev. **D18**, 2065 (1978).

[10] Y. Okawa and H. Ooguri, "Energy-momentum tensors in matrix theory and non-commutative gauge theories", hep-th/0103124.

[11] J. Barcelos-Neto, "Non-commutative fields in curved space", hep-th/0212094.